

LINEAR CHARACTERS AND BLOCK ALGEBRA

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ABSTRACT. This paper will prove that: 1. G has a block only having linear ordinary characters if and only if G is a p -nilpotent group with an abelian Sylow p -subgroup; 2. G has a block only having linear Brauer characters if and only if $O_{p'}(G) \leq O_{p'p}(G) = HO_{p'}(G) = \text{Ker}(B_0^*) \leq O_{p'pp'} = G$, where $H = G' O_{p'}(G)$, $\text{Ker}(B_0^*) = \bigcap_{\lambda \in \text{IBr}(B_0)} \text{Ker}(V_\lambda)$, B_0 is the principal block of G and V_λ is the $F[G]$ -module affording the Brauer character λ ; 3. if G satisfies the conditions above, then for any block algebra B of G , we have

$$\frac{\text{Dim}_F(B)}{|D|} = \sum_{\varphi \in \text{IBr}(B)} \varphi(1)^2$$

where D is the defect group of B .

INTRODUCTION

Let G be a finite group. Let $\text{Irr}(G)$ and $\text{IBr}(G)$ denote the set of irreducible ordinary characters and irreducible Brauer characters of G , respectively. We use $\text{cd}(G)$ to denote the set $\{\chi(1) | \chi \in \text{Irr}(G)\}$. The classical result asserts that G has a normal abelian p -complement if and only if $\text{cd}(G)$ is a set of p -power. J. Thompson[1] proved that G has a normal p -complement if p divides every element $\chi(1) > 1$ in $\text{cd}(G)$. G. Navarro and P. H. Tiep [2] improved this result by considering the fields of character values. Let B be a p -block algebra of G and $\text{cd}(B) = \{\chi(1) | \chi \in \text{Irr}(B)\}$. M. Isaacs and S.D.Smith[3] proved that G is p -nilpotent if p divides every element (> 1) in $\text{cd}(B_0)$, where B_0 is the principal block of G . G. Navarro and G. Robinson[4] proved that B is a nilpotent block if $\text{cd}(B)$ is a set of p -power, using the Classification of Finite Simple Groups.

Now let $\text{bcd}(G) = \{\varphi(1) | \varphi \in \text{IBr}(G)\}$. It is easy to prove that G has a normal Sylow p -subgroup P and G/P is abelian if and only if $\text{bcd}(G) = \{1\}$. A theorem of Michler[5] asserts G has a normal Sylow p -subgroup if and only if there is no element in $\text{bcd}(G)$ divided by p . P. H. Tiep and W. Willems[6] considered another extreme case when $\text{bcd}(G)$ is a set of a power of a fixed prime number s . They proved G is solvable unless $p = s = 2$.

For a p -block B of G , we define $\text{bcd}(B) = \{\varphi(1) | \varphi \in \text{IBr}(B)\}$. Generally, it is very difficult to get the information of G by using the set $\text{bcd}(B)$ or $\text{cd}(B)$ (see the articles mentioned above). In this paper, we want to decide the group G under the condition of existing some $\text{bcd}(B) = \{1\}$ or $\text{cd}(B) = \{1\}$. We succeed in finding all these groups satisfying our condition (see Theorem 2.4). One typical case is

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$S_4, p = 3$. Our ideal mainly comes from two papers, one from G. Chen[7] and another from X.Y. Chen, etc.[8]. G. Chen proved a result on zeros of ordinary characters, and X. Y. Chen, etc. generalized the result to Brauer characters. In this paper, we will improve their results(see Corollary 1.4).

T. Holm and W. Willems[9] presented some conjectures on Brauer character degrees. One of their conjectures is in block form: let B be a block of G with defect group D , is it true the following:

$$\frac{\text{Dim}_F(B)}{l(B)|D|} \leq \sum_{\varphi \in \text{IBr}(B)} \varphi(1)^2 ?$$

with the equality if and only if $l(B) = 1$, where $l(B) = |\text{IBr}(B)|$. They have proved the conjecture holds for p -solvable groups. Our results show that the groups in our cases are p -solvable and furthermore we will get:

$$\frac{\text{Dim}_F(B)}{|D|} = \sum_{\varphi \in \text{IBr}(B)} \varphi(1)^2$$

for any $B \in \text{Bl}(G)$, if G satisfies our condition.

For most results in Character theory and Brauer character theory, refer to [10][11][12][13].

Section 1 will define an action of linear Brauer characters on $\text{IBr}(G)$ and set of projective characters, giving results on zero points of Brauer characters. Section 2 will define an action of linear Brauer characters on the set of block algebras of G , giving our main results in this paper.

The following notation and terminology will be used throughout in this paper. G is a finite group. $p > 0$ is a prime number. (K, R, F) is a splitting p -module system, that means R is a complete discrete valuation ring with a maximal ideal π such that $F = R/\pi$ is a field of characteristic p and K is the quotient field of R . A p -regular element means an element of G whose order is prime to p . G^0 is the set of all p -regular elements in G . $\text{Bl}(G)$ denotes the set of p -blocks of G . B_0 denotes the principal block of G , containing the trivial character. We use $\text{LBr}(G)$ to denote the set of all linear Brauer characters of G .

1. ACTION OF $\text{LBr}(G)$ ON $\text{IBr}(G)$

Let G be a finite group and let $\text{IBr}(G)$ be the set of irreducible Brauer characters of G . For a $\varphi \in \text{IBr}(G)$, We use V_φ to denote the $F[G]$ -module affording the Brauer character φ . Let $\text{LBr}(G)$ denote the set of all linear Brauer characters of G . Take $\lambda \in \text{LBr}(G)$ and $\varphi \in \text{IBr}(G)$, then $\lambda\varphi$ is a Brauer character afforded by $V_\lambda \otimes V_\varphi$, which is also irreducible $F[G]$ -module. Hence $\lambda\varphi \in \text{IBr}(G)$. If V is a $F[G]$ (right)-module, we define

$$V^* = \text{Hom}_F(V, F)$$

as $F[G]$ -module by $fx(u) = f(ux^{-1}), x \in G, u \in V$, for $f \in \text{Hom}_F(V, F)$. V^* affords Brauer character φ^* such that $\varphi^*(x) = \varphi(x^{-1})$, if V affords character φ .

It follows from discussion above that $\text{LBr}(G)$ is an abelian group and has an action on the set $\text{IBr}(G)$.

Let P_φ be the projective cover of $\varphi \in \text{IBr}(G)$. Since $\lambda\varphi \in \text{IBr}(G)$ for $\lambda \in \text{LBr}(G)$, what can we say about $P_{\lambda\varphi}$?

Proposition 1.1. *For $\lambda \in \text{LBr}(G)$, $\varphi \in \text{IBr}(G)$, let V_φ and P_φ be the corresponding simple module and projective cover, respectively. Then we have $P_{\lambda\varphi} = V_\lambda \otimes P_\varphi$.*

Proof. We have

$$P_\varphi \longrightarrow V_\varphi \longrightarrow 0.$$

Then

$$V_\lambda \otimes P_\varphi \longrightarrow V_\lambda \otimes V_\varphi \longrightarrow 0.$$

Since $V_\lambda \otimes P_\varphi$ is projective and indecomposable, it follows that $P_{\lambda\varphi} = V_\lambda \otimes P_\varphi$ and the proof is complete. \square

Remark 1.2. Let Φ_φ denote the Brauer character afforded by P_φ for $\varphi \in \text{IBr}(G)$. Let $\text{PIBr}(G) = \{\Phi_\varphi | \varphi \in \text{IBr}(G)\}$. By Proposition 1.1, the group $\text{LBr}(G)$ has an action on $\text{PIBr}(G)$ by $\lambda\Phi_\varphi = \Phi_{\lambda\varphi}$.

Let $\{\Delta_i | i = 1, 2, \dots, n\}$ be the set of orbits of $\text{LBr}(G)$ acting on $\text{IBr}(G)$. Hence

$$\text{IBr}(G) = \bigcup_{i=1, \dots, n} \Delta_i.$$

Take a sum

$$\sigma = \sum_{\varphi \in \Delta} \varphi,$$

where Δ is an orbit of $\text{LBr}(G)$ acting on $\text{IBr}(G)$. Then we have $\lambda\sigma = \sigma$, for all $\lambda \in \text{LBr}(G)$. If $g \in G_{p'}$ is a p -regular element and $\sigma(g) \neq 0$, then

$$g \in \bigcap_{\lambda \in \text{LBr}(G)} \text{Ker}(\lambda) = G'O^{p'}(G),$$

where $O^{p'}(G)$ is a subgroup of G generated by all p -elements.

Notation: G^0 is the set of p -regular elements of G . Δ_φ is the orbit containing φ under the action of $\text{LBr}(G)$ on $\text{IBr}(G)$. Φ_φ denotes the Brauer character of projective cover of φ .

By using the discussions above it can be shown the following result:

Theorem 1.3. *Let $\varphi \in \text{IBr}(G)$ and $x \in G^0 - H^0$, where $H = G'O^{p'}(G)$. Then the order $o(xH) | |\Delta_\varphi|$ or both $\varphi(x) = 0$ and $\Phi_\varphi(x) = 0$.*

Proof. Let T_φ be the subgroup of $\text{LBr}(G)$ that fixes φ . Then $\Delta_\varphi = \{\lambda_i\varphi | \lambda_i \in \text{LBr}(G)/T_\varphi\}$. Let $\rho = \prod \lambda_i\varphi$ and we denote $|\Delta_\varphi|$ by t . Hence we have $\lambda^t\rho = \rho$ for any $\lambda \in \text{LBr}(G)$. If $\rho(x) \neq 0$ then $\lambda(x^t) = 1$. Hence $x^t \in H$ and $o(xH) | t$. If $\rho(x) = 0$ then we get $\varphi(x) = 0$.

By Proposition 1.1, ρ has the projective cover with Brauer character $\Phi_\rho = \prod \lambda_i\Phi_\varphi$. Hence $\lambda^t\Phi_\rho = \Phi_\rho$. It follows from the same reason as above that $o(xH) | |\Delta_\varphi|$ or $\Phi_\varphi(x) = 0$. \square

Let l denote the number of irreducible Brauer characters. Since $l = \sum |\Delta_\varphi|$ this implies the existence of orbit Δ_φ such that $o(xH)$ does not divide $|\Delta_\varphi|$, if $(l, o(xH)) = 1$. Hence we have the following result as a generalization of [8].

Corollary 1.4. For $x \in G^0 - H^0$. Then we have

- (1) If $o(xH)$ does not divide l , then there exists $\varphi \in \text{IBr}(G)$ such that $\varphi(x) = 0$ and $\Phi_\varphi(x) = 0$.

- (2) If $(o(xH), l) = 1$, then there exists $\varphi \in \text{IBr}(G)$ such that $\varphi(x) = 0$ and $\Phi_\varphi(x) = 0$.

Proof. This is an immediate consequence of Theorem 1.3. \square

2. ACTION OF $\text{LBr}(G)$ ON $\text{Bl}(G)$

$\text{Bl}(G)$ denote the set of block algebra of $F[G]$.

For $B \in \text{Bl}(G)$ and $\lambda \in \text{LBr}(G)$, we define $\text{IBr}(\lambda B) = \{\lambda\varphi \mid \varphi \in \text{IBr}(B)\}$. Choose any two elements $\phi, \varphi \in \text{IBr}(B)$. Then

$$(2.1) \quad (\Phi_{\lambda\varphi}, \Phi_{\lambda\phi}) = (\Phi_\varphi, \Phi_\phi)$$

and

$$(2.2) \quad (\lambda\varphi, \lambda\phi)_{G^0} = (\varphi, \phi)_{G^0}$$

by which we have that $\lambda\phi \in \text{IBr}(\lambda B)$ if and only if $\phi \in \text{IBr}(B)$. Hence $\lambda B \in \text{Bl}(G)$. This defines an action of $\text{LBr}(G)$ on $\text{Bl}(G)$. By (2.1) and (2.2), the action keeps inner product of characters. Hence we have

Proposition 2.1. *Let $\lambda \in \text{LBr}(G)$ and $B \in \text{Bl}(G)$. Then there is a inner-product-preserving bijection from B to λB under the map: $\varphi \mapsto \lambda\varphi, \Phi_\varphi \mapsto \lambda\Phi_\varphi = \Phi_{\lambda\varphi}$. In particular, B and λB has the same Cartan matrix.*

Let B_0 denote the principal block of G , namely the unique one which contains trivial character 1.

Theorem 2.2. *Let B be a block of G containing linear character $\lambda \in \text{LBr}(G)$. Then B is isomorphic to B_0 in the means of Proposition 2.1.*

Proof. Since $\lambda \in \text{LBr}(B)$, we have $B = \lambda B_0$. By Proposition 2.1, the result follows. \square

A wellknown fact is that all of irreducible Brauer characters are linear if and only if G has a normal p -subgroup P and G/P is abelian. Now if G has a block containing only linear irreducible Brauer characters, how can we say about the group G ?

Suppose $B \in \text{Bl}(G)$ and $\text{IBr}(B) \subseteq \text{LBr}(G)$.

By Theorem 2.2, it means the principal block B_0 having just linear irreducible Brauer characters.

Let

$$\text{Ker}(B) = \bigcap_{\chi \in \text{Irr}(B)} \text{Ker}(\chi), \text{Ker}(B^*) = \bigcap_{\varphi \in \text{IBr}(B)} \text{Ker}(V_\varphi).$$

Then we know

$$\text{Ker}(B) = O_{p'}(\text{Ker}(\chi)), \chi \in \text{Irr}(B)$$

and

$$\text{Ker}(B^*)/\text{Ker}(B) = O_p(G/\text{ker}(B))$$

If $B = B_0$ is the principal block, then

$$\text{Ker}(B_0) = O_{p'}(G)$$

and

$$\text{Ker}(B_0^*)/O_{p'}(G) = O_p(G/O_{p'}(G))$$

Suppose that B_0 contains only linear Brauer irreducible characters and $l(B_0) > 1$ (If $l(B_0) = 1$, then G is p -nilpotent). Let $1 \neq \lambda \in \text{IBr}(B_0)$. Now let $H =$

$G'Op'(G)$. Since $H \leq \text{Ker}(\lambda)$, we can see λ as an ordinary irreducible character of G by the following way:

$$\text{IBr}(G/H) = \text{Irr}(G/H) \subseteq \text{Irr}(G).$$

Then

$$O_{p'}(\text{Ker}(\lambda)) = O_{p'}(G).$$

In another word, we may consider $\text{IBr}(B_0) \subseteq \text{Irr}(B_0)$.

Then

$$H \leq \text{Ker}(B_0^*) = \bigcap_{\lambda \in \text{IBr}(B_0)} \text{Ker}(V_\lambda) = \bigcap_{\lambda \in \text{IBr}(B_0)} \text{Ker}(\lambda).$$

Now by using discussion above, we have the following result:

Theorem 2.3. *Let $H = G'Op'(G)$. Let G has a block B such that $\text{IBr}(B) \subseteq \text{LBr}(G)$. Then $\text{Ker}(B_0^*) = HO_{p'}(G)$.*

Proof. By the previous arguments, the case will be changed to consider the principal block B_0 satisfying our condition. Since G/H is a p' -group, $\text{Ker}(B_0^*)/HO_{p'}(G)$ must be a p' -group. The group $\text{Ker}(B_0^*)/HO_{p'}(G)$ is in fact a p -group since $\text{Ker}(B_0^*)/O_{p'}(G)$ is a p -group. Hence $\text{Ker}(B_0^*)/HO_{p'}(G) = 1$. \square

The following is our main result in this paper.

Theorem 2.4. *Let G be a finite group and p be a prime number. Let $H = G'Op'(G)$. Then*

- (1) *G has a p -block B such that $\text{Irr}(B) \subseteq \text{Lrr}(G)$ if and only if G is a p -nilpotent group with an abelian Sylow p -group;*
- (2) *G has a p -block B such that $\text{IBr}(B) \subseteq \text{LBr}(G)$ if and only if $O_{p'}(G) \triangleleft O_{p'p'}(G) = HO_{p'}(G) = \text{Ker}(B_0^*) \triangleleft O_{p'p'}(G) = G$.*

Proof. (1). If G is p -nilpotent group with an abelian Sylow p -group, we just check the principal block satisfying $\text{Irr}(B_0) \subseteq \text{Lrr}(G)$. Since $G' \leq O_{p'}(G) = \text{Ker}(B_0) \leq \text{Ker}\chi, \chi \in \text{Irr}(B_0)$, we have $\text{Irr}(B_0) \subseteq \text{Lrr}(G)$. Suppose a block B of G satisfies $\text{Irr}(B) \subseteq \text{Lrr}(G)$. Then we have $\text{IBr}(B) \subseteq \text{LBr}(G)$ and $l(B) = 1$ by using the decomposition matrix and Cartan matrix of B . By Theorem 2.2, the principal block B_0 has the same properties. Hence G is p -nilpotent. From $\text{Irr}(B_0) \subseteq \text{Lrr}(G)$, it follows that $G' \leq \bigcap_{\chi \in \text{Irr}(B_0)} \text{Ker}(\chi) = O_{p'}(G)$. Hence the Sylow p -subgroup must be abelian.

(2) The “only if” part is from the theorem 2.3. If $HO_{p'}(G) = \text{Ker}(B_0^*)$, then $\text{IBr}(B_0) \subseteq \text{IBr}(G/H) = \text{Irr}(G/H) = \text{Lrr}(G/H) = \text{LBr}(G/H) = \text{LBr}(G)$. Hence the “if” part is proved. \square

The following result is related to Holm and Willem’s local conjecture on Brauer character degrees. Of course, our result could contains much more information of group G , as it connects several block invariants in an equality.

Theorem 2.5. *If G has a block B satisfying $\text{IBr}(B) \subseteq \text{LBr}(G)$, then for any $B \in \text{Bl}(G)$, there is*

$$\frac{\text{Dim}_F(B)}{|D_B|} = \sum_{\varphi \in \text{IBr}(B)} \varphi(1)^2$$

where D_B denote the defect group of B . Hence Holm and Willems’ conjecture holds for these groups.

Proof. By Theorem 2.4, in G there is a sequence of normal subgroups:

$$O_{p'}(G) \trianglelefteq O_{p'p}(G) = HO_{p'}(G) \trianglelefteq O_{p'pp'}(G) = G$$

where $H = G'O^{p'}(G)$. Notice the group $HO_{p'}(G)$ is p -nilpotent, hence $l(b) = 1$ for any block $b \in \text{Bl}(HO_{p'}(G))$. Since $G/HO_{p'}(G)$ is a p' -group, we have $D_B \leq HO_{p'}(G)$. Our assertion follows from [14]. \square

Remark 2.6. In fact, Theorem 2.5 holds for all finite groups of p -length 1 which are defined in [3].

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